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ALL ORIENTABLE 2-MANIFOLDS HAVE FINITELY MANY MINIMAL TRIANGULATIONS

BY
D. W. BARNETTE AND ALLAN EDELSON
Department of Mathematics, University of California at Davis, Davis, CA 95616, USA

ABSTRACT
We show that for every orientable 2-manifold there is a finite set of triangulations from which all other triangulations can be generated by sequences of vertex splittings.

1. Introduction

One form of a well-known theorem of Steinitz [5] states that the triangulations of the 2-sphere can be generated from the complete graph on four vertices embedded in the 2-sphere, by a process called vertex splitting. Similar generation procedures have been found by Barnette [1] for the projective plane, generating the triangulations from two minimal triangulations; and by Grünbaum and Duke [2], Rusnak [4] and Lavrenchenko [3] for the torus, generating the triangulations from a set of 22 minimal triangulations. In this paper we show that for every orientable 2-manifold there is a finite set of minimal triangulations from which all others can be generated by vertex splitting.

2. Definitions

By a *2-manifold* we shall always mean a compact orientable 2-dimensional manifold.

If e is an edge of a triangulation T of a 2-manifold we say that T' is obtained from T by *edge shrinking* if T' can be obtained from T by removing e and all edges meeting e, and replacing them by a vertex v which is joined to every

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remaining vertex that was joined to a vertex of e. This process can most easily be visualized by imagining that the edge e is shrunk to the vertex v and any double edges produced are merged into single edges. We say that the edge e is *shrinkable*.

If T' is obtained from T by shrinking edge e to vertex v we also say that T is obtained from T' by *splitting* vertex v.

By a 3-*circuit* in a triangulation we mean the union of three edges e_1, e_2 and e_3 such that their pairwise intersections are three distinct vertices. If a 3-circuit does not bound a cell in M we say that it is a *nonplanar* 3-circuit, all other 3-circuits will be called *planar* 3-circuits.

If x and y are two vertices of an edge e we shall denote e by xy. If x, y and z are vertices of a 3-circuit C we denote C by xyz.

3. Minimal triangulations

If every edge of a triangulation T of a 2-manifold M is not shrinkable we say that T is a *minimal triangulation* of M. Clearly, if an edge e belongs to a nonplanar 3-circuit then e is not shrinkable. Thus if each edge belongs to a nonplanar 3-circuit then T is minimal. We begin by showing that the converse is also true for all 2-manifolds except the sphere.

LEMMA 1. *If T is a minimal triangulation of a manifold M, other than the sphere, then every edge of T lies on a nonplanar 3-circuit.*

PROOF. By a lemma of one of the authors [1], the edge e will be nonshrinkable if and only if it belongs to a 3-circuit that is not a triangle of the triangulation, thus an edge that is nonshrinkable and does not belong to a nonplanar 3-circuit must belong to a planar 3-circuit that is not a triangle of T.

In this case let C be a planar 3-circuit in M, that is not a triangle of T, but bounds a cell A, and let C be chosen to enclose a minimum number of vertices of T in the cell A. Let v be a vertex in A and not on C. No 3-circuit containing an edge e_1 meeting v can have any edge outside A because then T would have double edges. But, since e_1 is nonshrinkable, e_1 must lie in a 3-circuit that is not a triangle of T and lies in A, thus the minimality of C is violated. □

In the main theorem we will need to put a bound on the size of sets of homotopically different simple circuits in triangulations of orientable manifolds. This will be done in the following lemmas and Theorem 1. Let M be a 2-manifold of genus g, and let $\{e_1, \ldots, e_E\}$ be homotopically non-trivial closed curves at a base point v in M. Assume that the e_i satisfy the hypotheses

(H_1) $e_i \cap e_j = \{v\}$, for $i \neq j$,
(H_2) e_i is not homotopic to e_j, for $i \neq j$,
(H_3) the set $\{e_i\}$ is maximal with respect to H_1 and H_2.

The curves $\{e_i\}$ define a graph on M with regions $\{f_1, \ldots, f_F\}$, edges $\{e_1, \ldots, e_E\}$, and one vertex v.

LEMMA 2. *If* (H_1)–(H_3) *are satisfied then each edge* e_i *belongs to the boundary of exactly two faces.*

PROOF. It is obvious that each edge belongs to the boundary of at most two faces. We must show that e_i cannot be in the boundary of only one face. If M has genus $g = 0$ or $g = 1$, then the lemma is obvious. Assume $g > 1$ and that e_{i_1} belongs to the boundary of exactly one face f_1. The boundary $\mathscr{H}_1$ is a union of edges,

$$\mathscr{H}_1 = e_{i_1} \cup e_{i_2} \cup \cdots \cup e_{i_q} \cup e_{k_1} \cup \cdots \cup e_{k_r},$$

where each of the edges e_{i_j} intersects only the boundary of f_1, and each of the edges e_{k_j} intersects the boundary of exactly two faces. Since M is orientable each of the edges e_{k_j} has a closed tubular neighborhood N_j in $M - f_1$, which is homeomorphic to $e_{k_j} \times [0, 1]$. Then $M' = \bar{f}_1 \cup (\bigcup_{j=1}^{r} N_j)$ is a compact manifold with r boundary components $e'_{k_j} = e_{k_j} \times \{1\}$, for $j = 1, \ldots, r$. Assuming $r \geqq 1$, the manifold M' has genus $\tilde{g} < g$. Let $\tilde{M}$ be the closed manifold of genus $\tilde{g}$, obtained from M' by capping each of the boundary curves e'_{k_j}. The family of curves $e_{i_1}, e_{i_2}, \ldots, e_{i_q}$ on $\tilde{M}$ satisfy (i) and (ii), and by induction on the genus the family cannot be maximal. Therefore there exists a circuit γ at v such that $\gamma - v \subset f_1$, and the family $e_{i_1}, e_{i_2}, \ldots, e_{i_q}, \gamma$ satisfy (i) and (ii). We see that γ is not homotopic to any of the curves e_{k_j}, since this would imply γ is nullhomotopic in $\tilde{M}$. The lemma will be proved if we can show that γ is not homotopic in M to any of the curves $e_{i_j}, j = 1, 2, \ldots, q$. This will be accomplished by means of the Mayer–Vietoris homology sequence of the pair $U = \bar{f}_1$, $V = M - f_1$. This exact sequence is defined by

$$
\begin{aligned}
&\cdots \longrightarrow H_2(U \cup V) \xrightarrow{\Delta} H_1(U \cap V) \xrightarrow{\psi_*} H_1(U) \oplus H_1(V) \\
(1) \qquad &\xrightarrow{\phi_*} H_1(U \cup V) \xrightarrow{\Delta} H_0(U \cap V).
\end{aligned}
$$

Recall that the homomorphisms ψ_*, ϕ_* are induced by the chain maps

$$\psi : C_p(U \cap V) \to C_p(U) \oplus C_p(V),$$

$$\psi(c) = (c, -c),$$

$$\phi : C_p(U) \oplus C_p(V) \to C_p(U \cup V),$$

$$\phi(c^1, c^2) = c^1 + c^2.$$

In this case $U \cup V = M$, and $U \cap V = \mathscr{H}_1$, so the exact sequence becomes

$$(2) \qquad \begin{aligned} \cdots \longrightarrow H_2(M) \xrightarrow{\triangle} H_1(\partial \bar{f}_1) \xrightarrow{\psi_*} H_1(\bar{f}_1) \oplus H_1(M - f_1) \\ \xrightarrow{\phi_*} H_1(M) \xrightarrow{\triangle} H_0(\partial \bar{f}_1) \longrightarrow \cdots. \end{aligned}$$

Assume that γ is homotopic in M to e_{i_1}. Then $[\gamma] = [e_{i_1}]$ in $H_1(M)$, where [ ] denotes the homology class which is the image under the Hurewicz homomorphism. Therefore $\phi_*([\gamma], -[e_{i_1}]) = 0$, so by exactness $([\gamma], -[e_{i_1}])$ is in the image of ψ_*. The space $\partial \bar{f}_1$ is a wedge of circles, therefore the group $H_1(\partial \bar{f}_1)$ is free Abelian, with generators $\{[e_{i_1}], \ldots, [e_{i_q}], [e_{k_1}], \ldots, [e_{k_r}]\}$. Under the inclusion map $e_{i_1} \to \bar{f}_1$, the image of $[e_{i_1}]$ is a generator, therefore $([\gamma], -[e_{i_1}])$ cannot belong to the image of ψ_*. □

LEMMA 3. *If* (H_1)–(H_3) *are satisfied then each face* f_i *has boundary containing at least* 3 *edges.*

PROOF. Assume $\mathscr{H}_1$ contains exactly 1 edge, $\mathscr{H}_1 = e_1$. By the argument in the previous lemma the closure $\bar{f}_1$ is a compact surface with 1 boundary component, and with genus g_1 satisfying $0 \leqq g_1 < g$. If $g_1 = 0$ then $\bar{f}_1$ is homeomorphic to a disc D^2. In this case, e_1 is nullhomotopic in $\bar{f}_1$, hence it is nullhomotopic in M, contradicting the hypothesis. If $g_1 > 0$, there is a simple closed curve γ at v, such that $\gamma - v \subset f_1$, and γ is not homotopic to e_1 in $\bar{f}_1$. We will show that γ cannot be homotopic in M to any of the e_i, $1 \leqq i \leqq E$.

As in the previous lemma, we consider the Mayer–Vietoris sequence of the pair $U = \bar{f}_1$, $V = M - f_1$. In this case $H_1(U \cap V)$ is infinite cyclic with generator $[e_1]$, and the image of $[e_1]$ is one of the generators of $H_1(\bar{f}_1)$. As in the previous case $([\gamma], -[e_i])$ cannot belong to the image of ψ_*.

Now assume that $\mathscr{H}_1$ contains exactly two edges, $\mathscr{H}_1 = e_1 \cup e_2$. By the previous lemma each of the curves e_j, for $j = 1, 2$, has a closed, tubular neighborhood in M, homeomorphic to $e_j \times [-1, 1]$, with $e_j = e_j \times \{0\}$. Again by the previous lemma each edge belongs to exactly two faces so we can assume that $(e_j \times (0, 1]) \subset M - \bar{f}_1$, and that $(e_1 \times (0, 1]) \cap (e_2 \times (0, 1]) = \varnothing$. It follows that

$$\tilde{M} = f_1 \cup (e_1 \times [0, 1]) \cup (e_2 \times [0, 1])$$

is a compact manifold of genus $\tilde{g} < g$, having two boundary components $\tilde{e}_1 = e_1 \times \{1\}$, and $\tilde{e}_2 = e_2 \times \{1\}$. If $\tilde{g} = 0$, then $\tilde{e}_1 \simeq \tilde{e}_2$ in $\tilde{M}$, which implies $e_1 \simeq e_2$ in $\tilde{f}_1$, and hence also in M. If $\tilde{g} > 0$, then an algebraic proof based on the Mayer–Vietoris sequence, analogous to the previous case, contradicts the maximality of the edge set. □

THEOREM 1. *Let M be a 2-manifold of genus g, and let $\{e_1, \ldots, e_E\}$ be homotopically non-trivial closed curves at a base point v in M. Assume that the e_i satisfy hypotheses* (H_1)–(H_3). *Then*

$$E \leqq 6g - 3. \tag{3}$$

PROOF. As previously stated, the curves $\{e_i\}$ define a graph G on M with regions $\{f_1, \ldots, f_E\}$, edges $\{e_1, \ldots, e_E\}$, and one vertex v. From Eulers inequality

$$1 - E + F \geqq 2 - 2g. \tag{4}$$

The proof then follows by the usual counting arguments. In fact, by Lemma 2 we have $2E \geqq 3F$, which together with (4), implies $E \leqq 6g - 3$. □

LEMMA 4. *If F is a triangle of a triangulation T of an orientable 2-manifold M, then there is a bound depending only on the genus of M, on the number of homotopically different 3-circuits meeting any given vertex v of F.*

PROOF. Any two 3-circuits meeting v will meet just at v or on an edge containing v. Let S be a set of pairwise nonhomotopic 3-circuits through v. If all circuits in S meet only at v then we are done by Theorem 1. If some of them intersect on edges we contract those edges to the vertex v. Some of the 3-circuits now become 2-circuits, but they are still pairwise nonhomotopic and now Theorem 1 applies. □

If C_1 and C_2 are two homotopic 3-circuits in a triangulation T of a 2-manifold M then $C_1 \cup C_2$ will bound one of three types of subset of M. If $C_1 \cap C_2 = \varnothing$ then $C_1 \cup C_2$ bounds an annulus. If $C_1 \cap C_2$ is a vertex then $C_1 \cup C_2$ bounds a cell with two boundary points identified and if $C_1 \cap C_2$ is an edge e then $C_1 \cup C_2$ bounds a cell A together with the edge e meeting A only at its endpoints. Any of these three types of sets will be called *H-sets* for C_1 and C_2.

The following lemma will be used to establish a natural ordering on any finite set of pairwise homotopic 3-circuits.

LEMMA 5. *If $C = \{C_1, \ldots, C_n\}$ is a set of pairwise homotopic 3-circuits in a triangulation of a 2-manifold M then there exists an H-set A for a pair of circuits C_i and C_j such that all other circuits in C lie in A.*

PROOF. Our proof is by induction on n. The theorem is obvious if $n = 2$. If we have a set of k circuits we consider an H-set A_1 for C_1 and C_2. If not all of the remaining circuits lie in A_1 we contract A_1 to the circuit C_1. By induction there is an H-set A_2 for C_1 and the circuits not in A_1. If we now apply the inverse of our contraction we have an H-set containing all k 3-circuits. □

We now can describe a natural ordering of any finite set $\{C_1, \ldots, C_n\}$ of pairwise homotopic 3-circuits in a triangulation T of a 2-manifold M.

Let A be an H-set as generated by Lemma 5. We imbed A in the plane and define $C_i < C_j$ if and only if C_i has a vertex inside C_j. This is clearly a linear ordering on $\{C_1, \ldots, C_n\}$.

LEMMA 6. *If e is an edge in a triangulation T of a 2-manifold M and if e lies on four distinct nonplanar 3-circuits, all homotopic to each other, then T has a shrinkable edge.*

PROOF. We order the set of 3-circuits homotopic to C. Let $C_1, \ldots, C_4$ be four consecutive circuits in this ordering. Let $e = xy$ and let F be a triangle of T whose interior lies between C_2 and C_3, and meets x. Let $e_1 = x_1x_2$ be the edge of F missing x. Now suppose e_1 is not shrinkable. Then e_1 belongs to a nonplanar 3-circuit $x_1x_2x_3$ or in a planar 3-circuit that is not a triangle of T. In the latter case the argument in Lemma 1 shows that T has a shrinkable edge. But since x_1 and x_2 lie in A and miss the boundary of A, x_3 lies in A and $x_1x_2x_3$ is planar.

□

LEMMA 7. *If v is a vertex in a triangulation T of a 2-manifold M and if there are four distinct nonplanar 3-circuits, all homotopic to each other and meeting pairwise only at v then T has a shrinkable edge.*

PROOF. The proof is essentially the same as the proof of Lemma 6. □

LEMMA 8. *Let C be a nonplanar 3-circuit in a minimal triangulation T of a 2-manifold M. If there are 28 3-circuits homotopic to C then there exist four pairwise disjoint 3-circuits homotopic to C.*

PROOF. As in the previous lemma, we order all of the 3-circuits homotopic to C. Let $C_1, \ldots, C_{28}$ be the first 28 of them.

If C_2 meets C_1 on an edge e_1, then $C_2, \ldots, C_{28}$ do not contain the other two edges of C_1. Furthermore, among C_3 and C_4 is a circuit that does not contain e_1 for otherwise we may apply Lemma 6 and get a shrinkable edge. It follows that among $C_1, \ldots, C_4$ are two circuits C_1 and C_i having no edges and at most one vertex v_1 in common.

Similarly, among $C_i, \ldots, C_{i+3}$ is a circuit C_j meeting C_i on at most one vertex, and among $C_j, \ldots, C_{j+3}$ is a circuit C_k meeting C_j on at most one vertex. If C_1, C_i, C_j and C_k all meet at v_1 then we can apply Lemma 7 and get a shrinkable edge, thus one of C_i, C_j and C_k misses C_1.

We now know that among $C_2, \ldots, C_{10}$ we have a circuit C_r missing C_1. Similarly among $C_{r+1}, \ldots, C_{r+9}$ is a circuit C_s missing C_r (and thus also missing C_1), and among $C_{s+1}, \ldots, C_{s+9}$ is a circuit C_t missing C_s. Thus among $C_1, \ldots, C_{28}$ are four pairwise disjoint 3-circuits. □

LEMMA 9. *If C is a nonplanar 3-circuit in a minimal triangulation T of a 2-manifold M, then no more than 27 3-circuits of T are homotopic to C.*

PROOF. Suppose 28 such circuits exist. We order the set S of all 3-circuits homotopic to C in the usual way and let C_1, C_2, C_3, C_4 be the first four 3-circuits in a maximal collection of pairwise disjoint 3-circuits, as guaranteed by Lemma 8.

Case I. Some circuit $B \in S$ lies between C_2 and C_3. Then B shares a vertex or edge with C_2 or C_3. Suppose, without loss of generality B meets C_2. We may suppose that B and C_2 are chosen such that they are consecutive in the ordering of S. Let x be a vertex on both B and C_2. Let F be a face of T meeting x and whose interior lies between C_2 and B. Let e be an edge of F missing x.

Since T is minimal, e lies on a nonplanar 3-circuit B'. The circuit B' must lie in the annulus bounded by C_1 and C_4 because the third vertex of the circuit must lie between two of $C_1, \ldots, C_4$. Since B' is nonplanar and lies in an annulus it is homotopic to one of the bounding circuits of the annulus and thus is homotopic to C. This contradicts the fact that B and C_2 were consecutive. Thus e is shrinkable, a contradiction.

Case II. No such circuit as B exists. Then C_2 and C_3 are consecutive in S. We choose any edge e lying in the annulus bounded by C_2 and C_3, but not on the boundary of this annulus. The argument in Case I shows that e is shrinkable. □

THEOREM 2. *The orientable 2-manifolds of any genus have finitely many minimal triangulations.*

PROOF. Let T be a minimal triangulation of a manifold M. If $g = 0$ the theorem is well known to be true, thus we assume $g > 0$. Now, every edge of T belongs to a nonplanar 3-circuit.

Let C be a nonplanar 3-circuit in T. We cut M along C producing either a single manifold with a boundary consisting of two circuits or two manifolds with boundary, each with one bounding circuit. We span each of these bounding circuits by cells producing one or two manifolds of lower genus. Let M_1 be one manifold thus produced (if the other exists let it be called M_2). From the triangulation T_1 of M_1 we produce a minimal triangulation by shrinking edges. We shall find a bound on the number of shrinkings necessary to do this.

Every edge of T_1 lies in a nonplanar 3-circuit in T, thus if it does not lie in a nonplanar 3-circuit in T_1 it either lies in a nonplanar 3-circuit of T that is planar in T_1 (and thus is homotopic to C) or it lies in a nonplanar 3-circuit that does not exist in T_1 because the circuit was separated at a vertex when C was cut. Such circuits must therefore meet a vertex of C.

In the first case, by Lemma 9, there are at most 27 such 3-circuits homotopic to C. Thus there are at most 81 shrinkable edges of T_1 on such circuits.

In the second case, by Theorem 1 there is a bound K, depending only on the genus of M, on the number of homotopically different 3-circuits meeting each vertex of T. There can thus be at most $27K$ nonplanar 3-circuits meeting each vertex of C in M by Lemma 9, and thus at most $81K$ nonplanar 3-circuits destroyed by cutting along C.

We now know that at most $243K + 81$ edges fail to lie on nonplanar 3-circuits in T. Since shrinking an edge does not change the homotopy type of a circuit, if an edge lies on a nonplanar 3-circuit before a shrinking, it does so after the shrinking. Thus we reduce T_1 to a minimal triangulation of M_1 with at most $243K + 81$ edge shrinkings.

We finish our proof by proceeding by induction. Our theorem is true when the genus is 0. By induction the theorem is true for M_1, thus there are finitely many minimal triangulations of M_1. By the above argument there are finitely many possible combinatorial types for the triangulation T_1 because it is obtained by at most $243K + 81$ vertex splittings to the minimal triangulations of T_1. Similarly the triangulation T_2 of M_2, if it exists, has finitely many combinatorial types.

Finally, there are only finitely many ways of identifying two triangles of T_1, or of gluing a triangle of T_1 to a triangle of T_2, thus there are finitely many minimal triangulations of M. □

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